The dotted lines show the manner in which sulphur might be separated and the catalyst regenerated.

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*Half-Integral Quantum Numbers in the Theory of the Stark Effect and a General Hypothesis of Fractional Quantum Numbers.**

By A. M. Mosharrafa, Ph.D., D.Sc.

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§ 1. Indications of the occurrence of fractional quantum numbers have already been found by Kratzer† from a study of certain band spectra, and by Curtis‡ in his experimental investigation of helium bands. Heisenberg§ has also employed half-integral numbers in a quantum theory of the anomalous Zeeman effect with considerable success. The fourth of the slightly extended quantum conditions recently suggested by W. Wilson|| and applied by the present writer¶ to the Zeeman effect involves a fractional quantum number, as O. W. Richardson** has pointed out.

K. F. Niessen,†† in his work on the positively ionised hydrogen molecule, observes certain discontinuities in the energy graphs between symmetrical and asymmetrical models which strongly suggest the occurrence of half-integral quantum orbits for the latter. He refrains, however, from making

* This paper summarises one portion of a Thesis approved for D.Sc. degree in the University of London.

† 'Sitz. d. Bayer. Akad.,' p. 107 (March, 1922).

‡ 'Roy. Soc. Proc.,' A, vol. 103, p. 321 (1923).

§ 'Zeitschr. f. Phys.,' vol. 8, p. 273 (1921–22).

|| 'Roy. Soc. Proc.,' A, vol. 102, p. 478 (1922).

¶ 'Roy. Soc. Proc.,' A, vol. 102, p. 529 (1923).

** 'Phil. Mag.,' vol. 46, p. 911 (1923).

†† "Zur Quantentheorie des Wasserstoffmolekülions" (J. Van Druten, Utrecht), p. 108 (1922).

this assumption, on the ground that it would necessitate the adoption of half numbers in the somewhat allied case of the Stark effect.

The aim of this paper is to draw attention to certain experimental and theoretical indications of the occurrence of half-integral numbers in the Stark effect and to the special merit of a general quantum hypothesis involving fractional numbers.

§ 2. From the point of view of the quantum theory, the choice of the particular set of Hamiltonian co-ordinates for a degenerate system (in the Schwarzschild sense) affects, as is well known, the orbital configurations determined. Hence, some criterion is necessary to determine the choice. Sommerfeld* maintains that the " proper " set is the one in which the variables are separate when the relativity change in mass is taken into account. If such a unique set is available, then a definite physical picture of the steady states of the system could be ascertained, otherwise the physical orbital configurations are indeterminable.

Consider now a system of particles which possesses a unique set of proper co-ordinates (p, q), and, consequently, a corresponding determinable ensemble of physical orbits Γ_p^i, under certain conditions, but *no* proper co-ordinates, in the Sommerfeld sense, under slightly altered conditions : as in the case of the Stark effect. If under the latter conditions the variables are separable on non-relativistic dynamics in the co-ordinates (P, Q), then only a portion of the total number of orbits Γ_P^i determined on (P, Q) will in general be such as to coincide with members of the ensemble Γ_p^i when the difference in conditions vanishes. Consequently, only this portion of orbits can be said to possess a physical reality. We shall, however, show that, in the case of the Stark effect, it is possible by adopting half-integral quantum numbers to increase the number of orbits among Γ_P^i which so possess a physical reality : in fact, to multiply it by 2. Moreover, it will be shown that the totality of coincident physical orbits thus obtained will be such as to comprise *all* the orbits for which the geometrical conditions allow of (P, Q), as well as (p, q) as proper co-ordinates.

We identify the system considered in the last paragraph with the hydrogen atom in the absence and presence of an external electric field F respectively. Here the Hamiltonian co-ordinates in the spherical polar system (r, θ, ψ) are the proper co-ordinates (denoted by p, q above) in the absence of the field ; whereas no proper co-ordinates are available in the presence of the field. In the parabolic system (ξ, η, ψ), however, the Hamiltonian co-ordinates (denoted by P, Q above) are separable on non-relativistic dynamics. The ensemble of

* ' Atombau und Spectrallinien,' English translation, p. 571 (1923).

physical orbits Γ_p^i defined in the absence of the field are given, on neglecting the relativity refinement, by the equations

$$\left.\begin{aligned} dt &= \frac{m_0 dr}{\sqrt{\alpha_1 + 2m_0 e\mathrm{E}/r - \alpha_2/r^2}} \\ 0 &= \frac{-dr}{r^2\sqrt{\alpha_1 + 2m_0 e\mathrm{E}/r - \alpha_2/r^2}} + \frac{d\theta}{\sqrt{\alpha_2 - \alpha_3/\sin^2\theta}} \\ 0 &= \frac{-d\theta}{\sin^2\theta\sqrt{\alpha_2 - \alpha_3/\sin^2\theta}} + \frac{d\psi}{\sqrt{\alpha_3}} \end{aligned}\right\} \quad (1)$$

where m_0 is the mass of the electron, $(-e)$ and E the respective charges on the electron and nucleus, and

$$\alpha_1 = 2m_0 \mathrm{W}, \quad (2)$$

W being the total energy.

The constants α_1, α_2, α_3 are defined by the quantum conditions :—

$$\left.\begin{aligned} \mathrm{J}_r &= \int_0 \sqrt{\alpha_1 + 2m_0 e\mathrm{E}/r - \alpha_2/r^2}\ dr = n_r h \\ \mathrm{J}_\theta &= \int_0 \sqrt{\alpha_2 - \alpha_3/\sin^2\theta}\ d\theta = n_\theta h \\ \mathrm{J}_\psi &= \int_0^{2\pi} \sqrt{\alpha_3}\, d\psi = n^\psi h \end{aligned}\right\} \quad (3)$$

On the parabolic system (ξ, η, ψ) if the field is assumed to vanish, we have, on the other hand, for the ensemble Γ_{P}^i the equations :—

$$\left.\begin{aligned} dt &= \frac{m_0\xi^2\, d\xi}{\sqrt{\alpha_1\xi^2 + 2m_0 e\mathrm{E} + \alpha_2' - \alpha_3/\xi^2}} + \frac{m_0\eta^2\, d\eta}{\sqrt{\alpha_1\eta^2 + 2m_0 e\mathrm{E} - \alpha_2' - \alpha_3/\eta^2}} \\ 0 &= \frac{d\xi}{\sqrt{\alpha_1\xi^2 + 2m_0 e\mathrm{E} + \alpha_2' - \alpha_3/\xi^2}} - \frac{d\eta}{\sqrt{\alpha_1\eta^2 + 2m_0 e\mathrm{E} - \alpha_2' - \alpha_3/\eta^2}} \\ 0 &= \frac{-d\xi}{\xi^2\sqrt{\alpha_1\xi^2 + 2m_0 e\mathrm{E} + \alpha_2' - \alpha_3/\xi^2}} \\ &\quad + \frac{d\eta}{\eta^2\sqrt{\alpha_1\eta^2 + 2m_0 e\mathrm{E} - \alpha_2' - \alpha_3/\eta^2}} + \frac{d\psi}{\sqrt{\alpha_3}} \end{aligned}\right\}, \quad (4)$$

where α_1, α_2', α_3 are defined by

$$\left.\begin{aligned} \mathrm{J}_\xi &= \int_0 \sqrt{\alpha_1\xi^2 + 2m_0 e\mathrm{E} + \alpha_2' - \alpha_3/\xi^2}\ d\xi = n_\xi h \\ \mathrm{J}_\eta &= \int_0 \sqrt{\alpha_1\eta^2 + 2m_0 e\mathrm{E} - \alpha_2' - \alpha_3/\eta^2}\ d\eta = n_\eta h \\ \mathrm{J}_\psi &= \int_0^{2\pi} \sqrt{\alpha_3}\, d\psi = n_\psi h \end{aligned}\right\}. \quad (5)$$

The values of α_1 and α_3 are identical on the two systems of co-ordinates (which we have anticipated in the notation), giving:

$$\left.\begin{aligned}\alpha_1 &= -\left(\frac{2\pi m_0 e\mathrm{E}}{\mathrm{J}_r+\mathrm{J}_\theta+\mathrm{J}_\psi}\right)^2 = -\left(\frac{2\pi m_0 e\mathrm{E}}{\mathrm{J}_\xi+\mathrm{J}_\eta+\mathrm{J}_\psi}\right)^2\\ \alpha_3 &= (\mathrm{J}_\psi/2\pi)^2\end{aligned}\right\}, \tag{6}$$

provided

$$n_r + n_\theta = n_\xi + n_\eta, \tag{7}$$

and n_ψ is the same, for the two corresponding orbits considered. Further, the constants α_2, α_2' are given by

$$\alpha_2 = (\mathrm{J}_\theta + \mathrm{J}_\psi)^2/(2\pi)^2, \tag{8}$$

$$\left.\begin{aligned}\alpha_2' &= (\partial\mathrm{S}/\partial\xi)^2 - 2m_0\xi^2\mathrm{W} - 2m_0 e\mathrm{E} + \mathrm{J}_\psi^2/(2\pi)^2\,\xi^2\\ &= -(\partial\mathrm{S}/\partial\eta)^2 + 2m_0\eta^2\mathrm{W} + 2m_0 e\mathrm{E} - \mathrm{J}_\psi^2/(2\pi)^2\,\eta^2\\ &= 2m_0 e\mathrm{E}\,(\mathrm{J}_\xi - \mathrm{J}_\eta)/(\mathrm{J}_\xi + \mathrm{J}_\eta + \mathrm{J}_\psi)\end{aligned}\right\}, \tag{9}$$

where S is the function of action. For the relation between the co-ordinates themselves we have

$$\left.\begin{aligned}\xi &= \sqrt{2r}\,\cos\theta/2\,; \quad r = \tfrac{1}{2}(\xi^2+\eta^2)\\ \eta &= \sqrt{2}r\,\sin\theta/2\,; \quad \theta = 2\tan^{-1}(\eta/\xi)\end{aligned}\right\} \tag{10}$$

From (3), (5), (6) and (8) it follows that

$$\left.\begin{aligned}2\mathrm{J}_\xi &= \mathrm{J}_r + \mathrm{J}_\theta + \pi\alpha_2'/\sqrt{-\alpha_1}\\ 2\mathrm{J}_\eta &= \mathrm{J}_r + \mathrm{J}_\theta - \pi\alpha_2'/\sqrt{-\alpha_1}\end{aligned}\right\} \tag{11}$$

and from (9):

$$\alpha_2' = \tfrac{1}{2}\left[\left(\frac{\partial\mathrm{S}}{\partial\xi}\right)^2 - \left(\frac{\partial\mathrm{S}}{\partial\eta}\right)^2\right] - m_0\mathrm{W}\,(\xi^2-\eta^2) + \tfrac{1}{2}\left(\frac{\mathrm{J}\psi}{2\pi}\right)^2\left(\frac{1}{\xi^2}-\frac{1}{\eta^2}\right), \tag{12}$$

so that on substituting in this last equation from (10) and on using

$$\begin{aligned}\partial\mathrm{S}/\partial\xi &= \frac{\partial\mathrm{S}}{\partial r}\cdot\frac{\partial r}{\partial\xi} + \frac{\partial\mathrm{S}}{\partial\theta}\cdot\frac{\partial\theta}{\partial\xi}\\ &= \sqrt{2r}\cos\theta/2\,\frac{\partial\mathrm{S}}{\partial r} - \sqrt{\frac{2}{r}}\sin\theta/2\,\frac{\partial\mathrm{S}}{\partial\theta}\\ \partial\mathrm{S}/\partial\eta &= \frac{\partial\mathrm{S}}{\partial r}\frac{\partial r}{\partial\eta} + \frac{\partial\mathrm{S}}{\partial\theta}\frac{\partial\theta}{\partial\eta}\\ &= \sqrt{2r}\sin\theta/2\,\frac{\partial\mathrm{S}}{\partial r} + \sqrt{\frac{2}{r}}\cos\theta/2\,\frac{\partial\mathrm{S}}{\partial\theta}\end{aligned}$$

we have

$$\alpha_2' = r\cos\theta\left[\left(\frac{\partial S}{\partial r}\right)^2 - 2m_0 W - \left(\frac{n_\psi h}{2\pi}\right)^2 \Big/ r^2\sin^2\theta\right] - 2\sin\theta\,\frac{\partial S}{\partial r}\,\frac{\partial S}{\partial\theta} - \frac{\cos\theta}{r}\left(\frac{\partial S}{\partial\theta}\right)^2$$

$$= -2\sin\theta\,\frac{\partial S}{\partial r}\,\frac{\partial S}{\partial\theta} + 2m_0 e\mathrm{E}\cos\theta - \frac{2\cos\theta}{r}\left(\frac{\partial S}{\partial\theta}\right)^2 - \frac{2\cos\theta}{r\sin^2\theta}\left(\frac{n_\psi h}{2\pi}\right)^2 \tag{13}$$

since

$$(\partial S/\partial r)^2 - 2m_0 W = \frac{2m_0 e\mathrm{E}}{r} - \frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2 \frac{1}{r^2\sin^2\theta}\left(\frac{n_\psi h}{2\pi}\right)^2.$$

As α_2' is constant throughout the motion, we obtain its value from (13) on putting $\partial S/\partial\theta = 0$, thus

$$\alpha_2' = 2m_0 e\mathrm{E}\cos\beta - \frac{2}{r_0}\frac{\cos\beta}{\sin^2\beta}\left(\frac{n_\psi h}{2\pi}\right)^2 \tag{14}$$

where β is the root of

$$\partial S/\partial\theta = \sqrt{[(\mathrm{J}_\theta + \mathrm{J}_\psi)^2/(2\pi)^2 - \mathrm{J}_\psi^2/(2\pi\sin\theta)^2]} = 0$$

so that

$$\sin\beta = \pm\,\mathrm{J}_\psi/(\mathrm{J}_\theta + \mathrm{J}_\psi)$$

and r_0 is the value of r at $\theta = \beta$.

And on substituting for β in (14) we get

$$\alpha_2' = \pm\,2m_0 e\mathrm{E}\sqrt{[1 - \mathrm{J}_\psi^2/(\mathrm{J}_\theta + \mathrm{J}_\psi)^2]} \times [1 - (\mathrm{J}_\theta + \mathrm{J}_\psi)^2/(2\pi)^2 m_0 e\mathrm{E}r_0] \tag{15}$$

(11) and (15) yield

$$\left.\begin{aligned} 2\mathrm{J}_\xi &= \mathrm{J}_r + \mathrm{J}_\theta \pm (\mathrm{J}_r + \mathrm{J}_\theta + \mathrm{J}_\psi)\left[1 - \frac{(\mathrm{J}_\theta + \mathrm{J}_\psi)^2}{(2\pi)^2 m_0 e\mathrm{E}r_0}\right]\sqrt{[1 - \mathrm{J}_\psi^2/(\mathrm{J}_\theta + \mathrm{J}_\psi)^2]} \\ 2\mathrm{J}_\eta &= \mathrm{J}_r + \mathrm{J}_\theta \mp (\mathrm{J}_r + \mathrm{J}_\theta + \mathrm{J}_\psi)\left[1 - \frac{(\mathrm{J}_\theta + \mathrm{J}_\psi)^2}{(2\pi)^2 m_0 e\mathrm{E}r_0}\right]\sqrt{[1 - \mathrm{J}_\psi^2/(\mathrm{J}_\theta + \mathrm{J}_\psi)^2]} \end{aligned}\right\} \tag{16}$$

(16) shows that if J_r, J_θ, J_ψ are integral multiples of h, then J_ξ, J_η will not in general be so for any given orbit. Consequently the two sets of orbits, $\Gamma_p^{\,i}$, $\Gamma_{\mathrm{P}}^{\,i}$ defined by equations [(1) + (3)] and [(4) + (5)] respectively, where n_r, n_θ, n_ψ, n_ξ, n_η are all assumed to be integers, are not identical. Consider, however, the class of orbits $\Gamma_{p,\mathrm{P}}^{\,i}$ which satisfy one or both of the two conditions :

$$1 - \mathrm{J}_\psi^2/(\mathrm{J}_\theta + \mathrm{J}_\psi)^2 = 0, \tag{17}$$

$$1 - (\mathrm{J}_\theta + \mathrm{J}_\psi)^2/(2\pi)^2 m_0 e\mathrm{E}r = 0. \tag{18}$$

We have from (16) for this class of orbits

$$\left.\begin{aligned} 2\mathrm{J}_\xi &= \mathrm{J}_r + \mathrm{J}_\theta \\ 2\mathrm{J}_\eta &= \mathrm{J}_r + \mathrm{J}_\theta \end{aligned}\right\}, \tag{19}$$

so that one-half of this class, viz. those orbits for which $(n_r + n_\theta)$ is even,* is common between Γ_p^i and Γ_P^i if all quantum numbers are assumed to be integers.

We next examine the geometrical meaning of the two conditions (17) and (18). From (17) we have

$$J_\theta = 0, \tag{20}$$

in other words, the orbits lie in the equatorial plane or else are merely straight line orbits. From (18) we have

$$\left.\begin{aligned} r &= (J_\theta + J_\psi)^2/(2\pi)^2 m_0 e\mathrm{E} \\ &= \text{constant} \end{aligned}\right\}, \tag{21}$$

in other words, the orbit is a circle† and

$$J_r = 0. \tag{22}$$

Hence for the class of orbits $\Gamma_{p,P}^i$ at least one of the two co-ordinates (r, θ) is constant, and consequently *the variables are separable for such orbits in either of the two systems of co-ordinates on relativistic mechanics.* We should, therefore, expect either system to furnish us with the unique physical picture of these orbits. It is seen from (19) that if we write

$$n_\xi = m/2 = n_\eta \tag{23}$$

for these paths, where m is an integer $(= \overline{n_r + n_\theta})$, this agreement between the two systems is effected. The orbits defined by (4) and (5) will now have in common with those defined by (1) and (3) exactly those orbits for which either system is a proper system in the Sommerfeld sense.

§ 3. *Application to the Resolution of Lines in the Stark Effect.*—Consider an orbit for which at least one of the two conditions (17), (18) is satisfied in the absence of an external field, and therefore for which

$$n_\xi = n_\eta = m/2. \tag{23}$$

If now an electric field is slowly introduced, then since the quantum integrals are adiabatic invariants, (23) will still hold for the corresponding path in the presence of the field. We should, therefore, expect to find components in the spectral resolutions corresponding to such half-integral numbers. It must, however, be remembered that the number and intensity of such components cannot be expected to be large, since the probability of the occurrence of a circular or equatorial path is in general small, and, further, that the occurrence of such a path with a fractional $m/2$ is half this small probability. It is

* Regarding nought as an even integer.

† Orbits calculated on non-relativistic dynamics, for which the condition (18) is satisfied for $r = r_0$ alone, have no corresponding physical reality, and are therefore excluded from $_{p,P}^{i}$.

also to be noted that this probability decreases as $n_\xi + n_\eta + n_\psi$ [$= n_r + n_\theta + n_\psi$] increases. Thus a circular or equatorial path ($n_r = 0$; $n_\theta = 0$) is less likely to occur for $n_r + n_\theta + n_\psi = 5$ (say) than for $n_r + n_\theta + n_\psi = 2$. Consequently, *half-integral components are to be expected much less in the case of the higher than of the lower members of the Balmer series.* And, finally, the least improbable half-integral paths for hydrogen are the two paths corresponding to

$$n_\xi = n_\eta = 1/2, \qquad n_\psi = 1, \tag{24}$$

which are, in fact, identical in the absence of a field with the circular and equatorial orbits corresponding in spherical polar co-ordinates to

$$n_r = 0, \quad n_\theta = 1, \quad n_\psi = 1,$$

and

$$n_r = 1, \quad n_\theta = 0, \quad n_\psi = 1$$

respectively.

Table I gives the full scheme of components for H_β calculated on the hypothesis (23), the resolution $\Delta\nu$ being given by

$$\left.\begin{aligned} &\Delta\nu = 3h\mathrm{FN}/8\pi^2 m_0 \mathrm{E} \\ \text{and} \quad &\mathrm{N} = (n_2 - n_1)(n_1 + n_2 + n_3) - (m_2 - m_1)(m_1 + m_2 + m_3), \end{aligned}\right\} \tag{25}$$

where F is the field intensity, and n_1, n_2, n_3 are written for n_ξ, n_η, n_ψ respectively.

TABLE I.

$n_1+n_2+n_3 = 2$. Hβ. $m_1+m_2+m_3 = 4$.

Parallel components : $m_3-n_3 = 0$.		Normal components : $m_3-n_3 = \pm 1$.	
$m_1\ m_2\ m_3 \longrightarrow n_1\ n_2\ n_3$.	N.	$m_1\ m_2\ m_3 \longrightarrow n_1\ n_2\ n_3$.	N.
2 0 2 → 0 0 2	8	1 0 3 → 0 0 2	4
1 1 2 → 0 0 2	0	2 0 2 → 0 1 1	10
3 0 1 → 0 1 1	14	2 0 2 → 1 0 1	6
3 0 1 → 1 0 1	10	1 1 2 → 0 1 1	2
2 1 1 → 1 0 1	6	2 1 1 → 0 0 2	4
2 1 1 → 0 1 1	2	3 0 1 → 0 0 2	12
3 0 1 → ½ ½ 1	12	2 0 2 → ½ ½ 1	8
2 1 1 → ½ ½ 1	4	1 1 2 → ½ ½ 1	0
1½ 1½ 1 → 0 1 1	2	½ ½ 3 → 0 0 2	0
1½ 1½ 1 → ½ ½ 1	0	1½ 1½ 1 → 0 0 2	0

In the tables transitions leading to negative values of W are omitted for brevity. These values are obtained by interchanging n_1, n_2 and m_1, m_2 simultaneously in every case. The components above the dotted line in Table I correspond to integral values of the quantum numbers only. Those below the line involve half-integers. The top part of the table is identical with Sommerfeld's* results. In the lower part the components of orders 4, 12, and 0, 8 are added to the p– and n– groups respectively. Now *it is these additional components which represent the exact deficiency of Sommerfeld's tables* when compared with Stark's results.* Table I is in excellent agreement with experiment. The only deviation occurs in the case of the p– components of order 14, which have not been observed. These arise from whole-number transitions.

Further, if the restriction $n_1 = n_2$ for half-integers is removed, it can easily be verified that no additional components arise: the new transitions ensuing merely give rise to components which already exist in Table I for each of the two respective types of polarization. It is, however, improbable that such free use of half-integers for n_1 and n_2 is permissible—first owing to the lack of any theoretical justification for it, and, secondly, because it would lead to a number of components in the case of the other members of the Balmer series, which are not likely to have all escaped observation.

It is seen, on the other hand, from Table I that the bracketed transitions in which fractional numbers occur only in the single set $(n_1, n_2, n_3) = (\frac{1}{2}, \frac{1}{2}, 1)$

TABLE II.

$(n_1, n_2, n_3) = (\frac{1}{2}, \frac{1}{2}, 1)$.

	H_α.		H_γ.		H_δ.	
	m_1 m_2 m_3.	N.	m_1 m_2 m_3.	N.	m_1 m_2 m_3.	N.
Parallel components	1 1 1	0	2 2 1	0	5 0 1	30
	2 0 1	6	3 1 1	10	4 1 1	18
			4 0 1	20	3 2 1	6
Normal components	1 0 2	3	2 1 2	5	4 0 2	24
			3 0 2	15	3 1 2	12
					2 2 2	0

* *Loc. cit.*, p. 291, Tables 29 and 30.

completely account for the additional components, which complete Sommerfeld's tables for H_β, referred to above. *This is in full accordance with the above theory*, since the two orbits defined by this set of numbers are more likely to occur than any other half-integral orbits. Table II gives the additional components for H_α, H_γ, H_δ respectively arising from transitions in which fractional numbers occur only in this single set, these being the most probable half-integral components to appear for the respective Balmer "lines." Needless to say, experimental investigation with a view to the detection of these fresh components would be highly desirable.

§ 4. The accumulation of evidence quoted in § 1, together with the results of the other two sections, suggests a general form*

$$J_i = \int_0 \pi_i \, dq_i = (m_i/\mu_i)h, \qquad i = 12, \ldots \tag{26}$$

for the quantum conditions, where m_i and μ_i are integers. Here

$$\pi_i = p_i + eA_i, \qquad i = 1, 2, \ldots \tag{26A}$$

the q's and p's being Hamiltonian co-ordinates, e the charge on the particle in question, and A the corresponding four-vector potential; the sign $\int_0$ denotes integration over the complete period of libration of the independent variable. *This hypothesis possesses the special merit of leading to the classical law of least action as a limiting case.* Thus from (26) we have

$$\Delta J_i = \Delta \int_0 \pi_i \, dq_i = \Delta\,(m_i/\mu_i)\, h, \tag{27}$$

and the classical laws corresponding to (26) and (27) respectively are

$$J_i = \int_0 \pi_i \, dq_i = C_i \tag{26'}$$

$$\delta J_i = \delta \int_0 \pi_i \, dq_i = 0 \tag{27'}$$

where C_1, C_2, ... are arbitrary constants. Now let m_i and μ_i each increase indefinitely, so that

$$\underset{\binom{m_i \longrightarrow \infty}{\mu_i \longrightarrow \infty}}{Lt} \; [m_i/\mu_i]\, h = C_i \tag{28}$$

then

$$\underset{\binom{m_i \longrightarrow \infty}{\mu_i \longrightarrow \infty}}{Lt} \; [\Delta\,(m_i/\mu_i)] = Lt \left[\frac{\mu_i \Delta m_i - m_i \Delta \mu_i}{\mu_i\,(\mu_i + \Delta\mu_i)}\right]$$

$$= 0,$$

* See W. Wilson, *loc. cit.*

provided that Δm_i and $\Delta \mu_i$ are both finite; thus

$$\underset{\left(\substack{m_i \longrightarrow \infty \\ \mu_i \longrightarrow \infty}\right)}{\mathrm{L}t} [\Delta \mathrm{J}_i] = 0, \tag{29}$$

and it is seen that the classical laws (26′) and (27′) now hold as limiting cases of the quantum laws (26) and (27) respectively. The question of the actual behaviour of the quantum integers m_i, μ_i in a specified problem and the way it is linked up with the classical laws of radiation still, however, remains open to investigation. In the case of the simple hydrogen atom in the absence of external fields there is little doubt that m_i/μ_i assumes integral values alone. The considerations put forward in this paper point out to half integers within the domain of the phenomena considered. Could it be that the general form (26) holds the key to the behaviour of more complicated atoms and molecules?

The Quantum Theory and the Dielectric Constant.

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Introduction.

The change of energy of an atom of hydrogen when submitted to an electric field has been calculated by Epstein.* If W denotes the total energy of an atom, then the change in energy Δ W, due to the field F, is given by

$$\Delta \mathrm{W} = -\frac{3h^2\mathrm{F}}{8\pi^2 m\mathrm{E}}(n_2 - n_1)(n_1 + n_2 + n_3) + \frac{17e^2\mathrm{F}^2}{(16\pi \mathrm{R}_\mathrm{H})^2 m}\mathrm{Z}(n_1, n_2, n_3),$$

where R_H is Rydberg's constant for hydrogen and

$$\mathrm{Z}(n_1, n_2, n_3) = (n_1+n_2+n_3)^6 \left\{1 - \frac{3}{17}\left(\frac{n_1 - n_2}{n_1+n_2+n_3}\right)^2 - \frac{9}{17}\left(\frac{n_3}{n_1+n_2+n_3}\right)^2\right\},$$

and n_1 and n_2 are parabolic quantum numbers and n_3 is the equatorial quantum number.

The first term in the expression for Δ W represents the potential energy of the natural doublet of the orbit, and, of course, disappears for a circular orbit ($n_1 = n_2 = 0$). It corresponds to the quantity $\frac{3}{2}a\varepsilon\mathrm{F}\cos\theta$ in the case of a Kepler orbit, for $\frac{3}{2}a\varepsilon$ is the distance of the mean-time position of the electron

* 'Ann. der Physik,' vol. 51, p. 184 (1916).